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Stability Optimization of Positive Semi-Markov Jump Linear Systems via Convex Optimization [★]

Chengyan Zhao ^a, Masaki Ogura ^a, Kenji Sugimoto ^a

^a*Division of Information Science, Nara Institute of Science and Technology, Ikoma, Nara 630-0192, Japan*

Abstract

In this paper, we study the problem of optimizing the stability of positive semi-Markov jump linear systems. We specifically consider the problem of tuning the coefficients of the system matrices for maximizing the exponential decay rate of the system under a budget-constraint. By using a result from the matrix theory on the log-log convexity of the spectral radius of nonnegative matrices, we show that the stability optimization problem reduces to a convex optimization problem under certain regularity conditions on the system matrices and the cost function. We illustrate the validity and effectiveness of the proposed results by using an example from the population biology.

Key words: Semi-Markov jump linear systems, positive systems, stability optimization, convex optimization, bet-hedging population

1 Introduction

Markov jump linear systems (Costa et al. 2013) is an important class of stochastic dynamical systems able to model abrupt changes in system parameters, and has applications in mobile robots (Bowling 2006), epidemic processes (Ogura & Preciado 2016), and networked control systems (Hespanha et al. 2007). Several important issues on Markov jump linear systems have been addressed in the literature, such as stability analysis, state filtering, and feedback control (see, e.g., Shi & Li (2015) and referenced therein). However, Markov jump linear systems have a restriction that the sojourn times at each mode must follow exponential distributions. This assumption is not necessarily satisfied in practice; a typical example arises in the modeling of system failures in the context of fault tolerant control systems, in which realistic probability density functions of failure rates take the bathtub shape that are well-explained by Weibull distributions (Johnson 1989).

One natural way to overcome this limitation is to allow the sojourn times to follow non-exponential distributions, which results in a broader class of stochastic

dynamical systems called *semi-Markov jump linear systems*. In this context, we can find in the literature a great amount of recent effort toward the analysis and control of the class of systems. Schioler et al. (2014) have presented sufficient conditions for the moment stability of semi-Markov jump linear systems. Huang & Shi (2012) derived linear matrix inequalities (LMIs) for the robust state-feedback control of semi-Markov jump linear systems. There are several LMI-based approaches for the further advanced types of control of semi-Markov jump linear systems (Chen et al. 2016, Wei et al. 2017, Jiang et al. 2018, Shen et al. 2018, Yan et al. 2019). It is worthwhile to remark that the generality of the class of non-exponential distributions causes intrinsic difficulties when studying semi-Markov jump linear systems. For example, Zhang et al. (2017) showed that the mean square stability of a discrete-time semi-Markov jump linear system is equivalent to the solvability of an infinite system of LMIs, which is in great contrast with the case of Markov jump linear systems (Costa et al. 2013). For this reason, it has been left as an open problem to understand what control problems for semi-Markov jump linear systems can be efficiently solved with a global optimality.

In this paper, we show that a class of budget-constrained stabilization problem for *positive* semi-Markov jump linear systems can be optimally solved via standard convex optimization. We specifically consider the problem of tuning the coefficients of the system matrices under

[★] This paper was not presented at any IFAC meeting. Corresponding author C. Zhao.

Email addresses: zhao.chengyan.za5@is.naist.jp (Chengyan Zhao), oguram@is.naist.jp (Masaki Ogura), kenji@is.naist.jp (Kenji Sugimoto).

a budget-constraint for maximizing the exponential decay rate of the system. By using the stability characterization of positive semi-Markov jump linear systems (Ogura & Martin 2014) and the log-log convexity result on the spectral radius of nonnegative matrices (Kingman 1961), we show that the optimal stabilization problem reduces to a convex optimization problem under certain regularity conditions on the system matrices and the cost function as well as the uniform boundedness of sojourn times. We illustrate the validity and effectiveness of the proposed results by using an example from population biology (Kussell & Leibler 2005).

This paper is organized as follows. In Section 2, we formulate the stabilization problem of positive semi-Markov jump linear systems and state the main result. The derivation of the main result is presented in Section 3. In Section 4, we illustrate the validity and effectiveness of the result with numerical simulations.

Notations

Let (Ω, M, P) be a probability space. The expected value of a random variable X on Ω is denoted by $E[X]$. Let \mathbb{R} , \mathbb{R}_+ , and \mathbb{R}_{++} denote the set of real, nonnegative, and positive numbers, respectively. Let $\mathbb{R}^{n \times n}$ denote the set of $n \times n$ matrices. The identity matrix of order n is denoted by I_n . We say that a real vector x is nonnegative, denoted by $x \geq 0$, if the entries of x are all nonnegative. We say that a square matrix is Metzler if the off-diagonal entries of the matrix A are nonnegative. We denote the spectral radius of A by $\rho(A)$. We define the entrywise logarithm of a vector $v \in \mathbb{R}_{++}^n$ by $\log[v] = [\log v_1, \dots, \log v_n]^\top$. The entrywise exponential operation $\exp[\cdot]$ is defined in the same manner.

2 Main result

Let us consider a parameterized family of switched linear systems of the form

$$\Sigma_\theta : \frac{dx}{dt} = A_{\sigma(t)}(\theta)x(t), \quad x(0) = x_0 \in \mathbb{R}^n, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, σ is a piecewise-constant function taking values in the set $\{1, \dots, N\}$, and $A_1(\theta), \dots, A_N(\theta) \in \mathbb{R}^{n \times n}$ are matrices parametrized by the parameter θ belonging to a set $\Theta \subset \mathbb{R}^\ell$. In this paper, we specifically focus on the class of positive semi-Markov jump linear systems defined as follows:

Definition 1 (Ogura & Martin (2014)) *Let $\theta \in \Theta$. We say that the system Σ_θ is a positive semi-Markov jump linear system if $x_0 \in \mathbb{R}_+^n$, the matrices $A_1(\theta), \dots, A_N(\theta)$ are Metzler, and σ is a semi-Markov process taking values in $\{1, \dots, N\}$ (Janssen & Manca 2006).*

For $t \geq 0$ and $x_0 \in \mathbb{R}_+^n$, we let $x(t; x_0)$ denote the trajectory of the system Σ_θ at time t and with the initial condition $x(0) = x_0$. This paper is concerned with the stability property of the system Σ_θ given as follows:

Definition 2 *Let $\theta \in \Theta$. The exponential decay rate of the system Σ_θ is defined by*

$$\gamma_\theta = - \sup_{x_0 \in \mathbb{R}_+^n} \limsup_{t \rightarrow \infty} \frac{\log E[\|x(t; x_0)\|]}{t}.$$

In this paper, we consider a budget-constrained stability optimization problem described as follows. Consider the situation where a limited amount of resource available is given for tuning the parameter θ to improve the stability of the system Σ_θ . We let the real function $C(\theta)$ to denote the cost for achieving a specific parameter θ . In this context, we formulate our stability optimization problem as follows:

Problem 3 (Budget-constrained stabilization)

Let a real number \bar{C} be given. Find the parameter $\theta \in \Theta$ such that the exponential decay rate γ_θ is maximized, while the budget constraint $C(\theta) \leq \bar{C}$ is satisfied.

To state our assumptions on the system matrices $A_i(\theta)$ and the cost function $C(\theta)$, we introduce monomials and posynomials (Boyd et al. 2007). We say that a function $F: \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{++}$ is a *monomial* if there exist $c > 0$ and real numbers a_1, \dots, a_n such that $F(v) = cv_1^{a_1} v_2^{a_2} \dots v_n^{a_n}$. Then, we say that a function F is a *posynomial* if F is a sum of finite monomials.

Assumption 4 *The following conditions hold true:*

- (1) *For each $k = 1, \dots, N$, there exists an $n \times n$ Metzler matrix M_k such that each entry of the matrix $A_k(\theta) - M_k$ is either a posynomial in θ or zero.*
- (2) *$C(\theta)$ is a posynomial in θ .*
- (3) *There exist posynomials $g_1(\theta), \dots, g_m(\theta)$ and positive constants $\bar{g}_1, \dots, \bar{g}_m$ such that*

$$\Theta = \{\theta \in \mathbb{R}^\ell : g_1(\theta) \leq \bar{g}_1, \dots, g_m(\theta) \leq \bar{g}_m\}.$$

- (4) *The sojourn times of the semi-Markov process σ are uniformly bounded, i.e., there exists $T > 0$ such that the sojourn times are less than or equal to T with probability one.*

To state the main result of this paper, we introduce the following notations. Let σ_d be the embedded Markov chain of σ (see, e.g., Janssen & Manca (2006)). For $i, j \in \{1, \dots, N\}$, let p_{ij} denote the transition probability of σ_d , i.e., let p_{ij} denote the probability that the discrete-time Markov chain σ_d transitions into state j from state i in one time step. Also, let h_{ij} denote the random variable representing the sojourn time of σ at the mode j after

jumping from the mode i . The following theorem shows that Problem 3 can be solved by convex optimization and is the main result of this paper.

Theorem 1 *Let $\bar{C} > 0$ be given. For each $\theta \in \Theta$ and $g > 0$, define $\mathcal{A}(\theta, g) \in \mathbb{R}^{(nN) \times (nN)}$ as the block matrix whose (i, j) -block is defined by*

$$[\mathcal{A}(\theta, g)]_{ij} = p_{ji} E[e^{(A_j(\theta) + gI)h_{ji}}] \in \mathbb{R}^{n \times n}. \quad (2)$$

Also, define the set $\log[\Theta] = \{\log[\theta]: \theta \in \Theta\} \subset \mathbb{R}^\ell$. Then, the following optimization problem is convex:

$$\begin{aligned} \text{minimize}_{u \in \log[\Theta], v \in \mathbb{R}} \quad & -v \end{aligned} \quad (3a)$$

$$\text{subject to} \quad \log \rho(\mathcal{A}(\exp[u], e^v)) \leq 0, \quad (3b)$$

$$\log C(\exp[u]) \leq \log \bar{C}, \quad (3c)$$

$$\log g_i(\exp[u]) \leq \log \bar{g}_i, \quad (3d)$$

$$i = 1, \dots, \ell.$$

Moreover, if $u = u^*$ solves the convex optimization problem (3), then $\theta = \exp[u^*]$ solves Problem 3.

3 Proof

In this section, we give the proof of Theorem 1. We first prepare a few lemmas for the proof. The first lemma gives a characterization of the exponential decay rate of the system Σ_θ in terms of the spectral radius of the matrix $\mathcal{A}(\theta, g)$ defined in the theorem.

Lemma 5 *Let $\theta \in \Theta$ and $g > 0$ be arbitrary. The exponential decay rate of Σ_θ satisfies $\gamma_\theta > g$ if and only if $\rho(\mathcal{A}(\theta, g)) < 1$.*

PROOF. Assume $\gamma_\theta > g$. Then, the positive semi-Markov jump linear system $dx/dt = (A_{\sigma(t)}(\theta) + gI)x(t)$ is exponentially mean stable. Therefore, by Theorem 2.5 in Ogura & Martin (2014), the matrix $\mathcal{A}(\theta, g)$ has a spectral radius less than one, as desired. The proof of the opposite direction can be proved in the same manner and, therefore, is omitted.

We then recall the following celebrated result by Kingman (1961). We say that an \mathbb{R}_{++} -valued function $f(x)$ is superconvex if $\log f(x)$ is convex.

Lemma 6 (Kingman (1961)) *Let $A: \mathbb{R}^\ell \rightarrow \mathbb{R}_+^{n \times n}$ be a function. Assume that each entry of A is either a superconvex function or the zero function. Then, the mapping $\mathbb{R}^\ell \rightarrow \mathbb{R}_{++}: x \mapsto \rho(A(x))$ is superconvex.*

We finally state the following lemma concerning the superconvexity of posynomials.

Lemma 7 (Boyd et al. (2007)) *Let $f: \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{++}$ be a posynomial. Then, the mapping $\mathbb{R}_{++}^n \rightarrow \mathbb{R}_{++}: u \mapsto f(\exp[u])$ is superconvex.*

Let us now prove Theorem 1.

PROOF OF THEOREM 1 Lemma 5 shows that the solution of Problem 3 is given by the following optimization problem:

$$\begin{aligned} \text{minimize}_{\theta \in \Theta, \gamma > 0} \quad & 1/\gamma \\ \text{subject to} \quad & \rho(\mathcal{A}(\theta, g)) \leq 1, \\ & C(\theta) \leq \bar{C}, \\ & g_i(\theta) \leq \bar{g}_i, \quad i = 1, \dots, \ell. \end{aligned} \quad (4)$$

Performing the variable transformations $u = \log[\theta]$ and $v = \log g$ as well as taking logarithms in the objective functions and constraints, we can equivalently reduce (4) into the optimization problem (3). To complete the proof, let us show the convexity of the optimization problem (3). The convexity of the constraints (3c) and (3d) is a direct consequence of the superconvexity of posynomials stated in Lemma 7. We need to show the convexity of the mapping

$$\log[\Theta] \times \mathbb{R} \rightarrow \mathbb{R}: (u, v) \mapsto \log \rho(\mathcal{A}(\exp[u], e^v)). \quad (5)$$

For each $k = 1, \dots, N$, we define the matrix function $\tilde{A}_k(\theta) = A_k(\theta) - M_k + gI$. Then, equation (2) shows that $[\mathcal{A}(\theta, g)]_{ij} = p_{ji} \int_0^T e^{(\tilde{A}_j(\theta) + M_j)\tau} f_{ji}(\tau) d\tau$, where f_{ji} denotes the probability density function of the sojourn time h_{ji} . This equation and the Lie-product formula $e^{A+B} = \lim_{K \rightarrow \infty} (e^{A/K} e^{B/K})^K$ for square matrices A and B (see, e.g., Cohen (1981)) yield that

$$\begin{aligned} [\mathcal{A}(\theta, g)]_{ij} &= p_{ji} \int_0^T \lim_{K \rightarrow \infty} \left(e^{\frac{\tau \tilde{A}_j(\theta, g)}{K}} e^{\frac{\tau M_j}{K}} \right)^K f_{ji}(\tau) d\tau \\ &= \lim_{K, L \rightarrow \infty} \Gamma_{ij}^{(K, L)}(\theta, g) \end{aligned}$$

where, for positive integers K and L , the $n \times n$ matrix $\Gamma_{ij}^{(K, L)}(\theta, g)$ is defined by

$$\Gamma_{ij}^{(K, L)}(\theta, g) = p_{ji} \sum_{\ell=1}^L \frac{T}{L} \left(e^{\frac{\ell T \tilde{A}_j(\theta, g)}{KL}} e^{\frac{\ell T M_j}{KL}} \right)^K f_{ji}(\ell T/L).$$

Therefore, if we define

$$\begin{aligned} \Gamma_{ij}^{(K, L, M)}(\theta, g) &= \\ p_{ji} \sum_{\ell=1}^L \frac{T}{L} \left(\sum_{m=0}^M \frac{1}{m!} \left(\frac{\ell T \tilde{A}_j(\theta, g)}{KL} \right)^m e^{\frac{\ell T M_j}{KL}} \right)^K f_{ji} \left(\frac{\ell T}{L} \right) \end{aligned}$$

then, by the definition of matrix exponentials, we obtain

$$[\mathcal{A}(\theta, g)]_{ij} = \lim_{K, L, M \rightarrow \infty} \Gamma_{ij}^{(K, L, M)}(\theta, g). \quad (6)$$

We show that each entry of the matrix $\Gamma_{ij}^{(K, L, M)}$ is either a posynomial in θ and g or zero. Since the matrix M_j is assumed to be Metzler (Assumption 4.1), the matrix $e^{\ell T \Gamma_j / KL}$ is nonnegative for all K and L . Also, each entry of the matrix $\tilde{A}(\theta, g)$ is either a posynomial or zero by Assumption 4.1. Since the set of posynomials is closed under additions and multiplications, each entry of the matrix power $(\ell T \tilde{A}_j(\theta, g) / KL)^m$ is either a posynomial of θ and g or zero as well. From the above observation, we conclude that each entry of the matrix $\Gamma_{K, L, M}(\theta, g)$ is a posynomial with the variables θ and g or zero.

Now, define the $(nN) \times (nN)$ matrix $\mathcal{A}^{(K, L, M)}(\theta, g)$ as the block matrix whose (i, j) -block equals $\Gamma_{ij}^{(K, L, M)}(\theta, g)$ for all $i, j \in \{1, \dots, N\}$. Then, by Lemmas 6 and 7, the mapping $(u, v) \mapsto \rho(\mathcal{A}^{(K, L, M)}(\exp[u], e^v))$ is superconvex. Since (6) shows that the mapping \mathcal{A} is a point-wise limit of the mapping $\mathcal{A}^{(K, L, M)}$, taking a limit preserves superconvexity, and the spectral radius operator $\rho(\cdot)$ is continuous, we obtain the convexity of the mapping (5). This completes the proof of convexity of the optimization problem (3), as desired.

4 Numerical example

In this section, we illustrate the effectiveness of the main result with an example that arises in the context of population biology (Kussell & Leibler 2005). To survive through time-varying environments caused by, e.g., temperature shifts, day-night cycles, and pH shifts, organisms often exhibit a variety of phenotypes. In this example, we consider a biological community with n phenotypes living in a randomly fluctuating environment with N possible environmental types. Let g_k^i denote the growth rate of phenotype i under environment k . In the meantime, an individual having phenotype i transforms itself to phenotype j at a constant rate which is represented by ω_k^{ij} . Let $x_i(t)$ denote the size of population having phenotype i at time t and $\sigma(t)$ denote the environment type at time t . Then, the growth of the population with phenotype i is described (Kussell & Leibler 2005) as

$$\Sigma : \frac{dx_i}{dt} = g_{\sigma(t)}^i x_i(t) + \sum_{j \neq i} \omega_{\sigma(t)}^{ji} x_j(t), \quad (7)$$

where $\omega_k^{ii} = -\sum_{j=1, j \neq i}^n \omega_k^{ij}$.

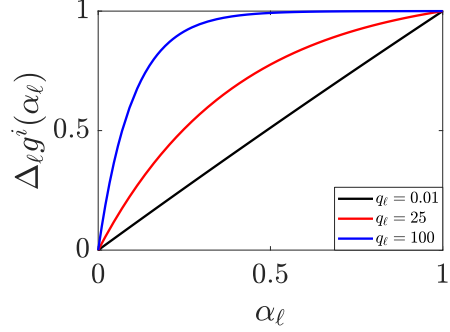


Fig. 1. Three realizations of the dosage-performance function with the parameter $\bar{\alpha}_\ell = 1, \underline{\theta}_\ell = 10, s_\ell^i = 1$ and $q_\ell = 0.01, 25, 100$.

Let us consider the problem of driving the entire population into extinction through medical intervention. Assume that L different types of antibiotics are available for suppressing growth rates. Let $c_\ell(\alpha_\ell)$ ($\ell \in \{1, \dots, L\}$) denote the cost for dosing α_ℓ unit of the ℓ th antibiotics, which is assumed to reduce the growth rate of the i th phenotype population by $\Delta_\ell g^i(\alpha_\ell)$ independent of the current environment types. In this situation, we can reduce the growth rate of the i th phenotype population to $g_{\sigma(t)}^i - \sum_{\ell=1}^L \Delta_\ell g^i(\alpha_\ell)$ with the associated total cost $C(\alpha) = \sum_{\ell=1}^L c_\ell(\alpha_\ell)$. The resulting population dynamics admits the representation

$$\Sigma' : \frac{dx_i}{dt} = \left(g_{\sigma(t)}^i - \sum_{\ell=1}^L \Delta_\ell g^i(\alpha_\ell) \right) x_i(t) + \sum_{j \neq i} \omega_{\sigma(t)}^{ji} x_j(t).$$

Let us allow the following box constraint

$$0 \leq \alpha_\ell \leq \bar{\alpha}_\ell \quad (8)$$

on the amount of doses. Under this scenario, we consider the following optimal intervention problem:

Problem 8 (Optimal intervention problem) *Let \bar{C} be a positive constant. Assume that σ is a semi-Markov process satisfying Assumption 4.4. Find the set of dose amounts $\alpha = (\alpha_1, \dots, \alpha_L)$ to maximize the exponential decay rate of the system Σ' while satisfying*

$$C(\alpha) \leq \bar{C}. \quad (9)$$

In this numerical example, we assume that the cost for antibiotics is linear with their dose amount, i.e., we let $c_\ell(\alpha_\ell) = r_\ell \alpha_\ell$ for a constant $r_\ell > 0$ for all ℓ . As for the suppression $\Delta_\ell g^i(\alpha_\ell)$ of the growth rates, we adopt the increasing function

$$\Delta_\ell g^i(\alpha_\ell) = s_\ell^i \frac{\underline{\theta}_\ell^{-q_\ell} - (\alpha_\ell + \underline{\theta}_\ell)^{-q_\ell}}{\underline{\theta}_\ell^{-q_\ell} - (\bar{\alpha}_\ell + \underline{\theta}_\ell)^{-q_\ell}}$$

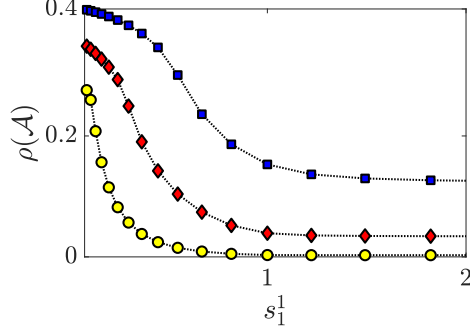


Fig. 2. The optimal spectral radius versus s_1^1 . Circles: $c_1 = 0.2$, triangles: $c_1 = 0.5$, squares: $c_1 = 1$.

where $\theta_\ell > 0$, $s_\ell^i \geq 0$, and $q_\ell > 0$ are parameters. These parameters allow us to realize various shapes of the suppression functions, including the dose-proportional suppression illustrated in Fig. 1. We notice that the zero dose of the ℓ th antibiotic does not change the growth rate, i.e., $\Delta_\ell g^i(\alpha_\ell)(0) = 0$, while the maximum dose achieves $\Delta_\ell g^i(\bar{\alpha}_\ell) = s_\ell^i$.

Let us show that the optimal intervention problem reduces to Problem 3. We introduce an auxiliary variable $\theta_\ell = \underline{\theta}_\ell + \alpha_\ell$ that is to be optimized. If we define $\bar{\theta}_\ell = \underline{\theta}_\ell + \bar{\alpha}_\ell$, then the constraint (8) is rewritten as the block constraint $\underline{\theta}_\ell \leq \theta_\ell \leq \bar{\theta}_\ell$, which can be expressed using posynomial functions (Preciado et al. 2014). Therefore, Assumption 4.3 is satisfied. Let us define the variable $\theta = (\theta_1, \dots, \theta_L)$. Then, we can rewrite the system Σ' into the form (1), where the matrices $A_1(\theta), \dots, A_N(\theta)$ are defined by $[A_k(\theta)]_{ii} = \tilde{g}_k^i + \sum_{\ell=1}^L s_\ell^i (\theta_\ell^{-q_\ell} - \bar{\theta}_\ell^{-q_\ell})^{-1} \theta_\ell^{-q_\ell}$ with $\tilde{g}_k^i = g_k^i - \omega_k^{ii} - \sum_{\ell=1}^L s_\ell^i \theta_\ell^{-q_\ell} / (\theta_\ell^{-q_\ell} - \bar{\theta}_\ell^{-q_\ell})$, and $[A_k(\theta)]_{ij} = \omega_k^{ij}$ for $i \neq j$. Therefore, if we define the diagonal matrix $M_k = \text{diag}(\tilde{g}_k^1, \dots, \tilde{g}_k^n)$, then each entry of the matrix $A_k(\theta) - M_k$ is a posynomial in the variables θ or zero. Hence, Assumption 4.1 is satisfied. Furthermore, the cost constraint (9) can be rewritten as $\sum_{\ell=1}^L r_\ell \theta_\ell \leq \bar{C} + \sum_{\ell=1}^L r_\ell \bar{\alpha}_\ell$ in terms of posynomials of the variable θ . Since all the conditions in Assumption 4 are satisfied, the optimal intervention problem can be efficiently solved by convex optimization as shown in Theorem 1.

For simplicity of presentation, we focus on the case of $n = N = 2$ in this numerical example. Throughout the simulation, we fix a part of the parameters as follows: $\omega_1^{21} = \omega_1^{12} = 0.1$, $\omega_2^{21} = \omega_2^{12} = 0.5$, $g_1^1 = 1$, $g_1^2 = -0.1$, $g_2^1 = -1$, $g_2^2 = 0.1$, $\bar{\alpha}_1 = \bar{\alpha}_2 = 1$, $\theta_1 = \theta_2 = 10$, $q_1 = q_2 = 0.01$, $s_1^2 = 0.4$, $s_2^1 = 1$, $s_2^2 = 0.4$, and $c_2 = 1$. Also, we assume that the environment keeps switching from one to another, and that the sojourn time of each environment follows the log-normal distribution having

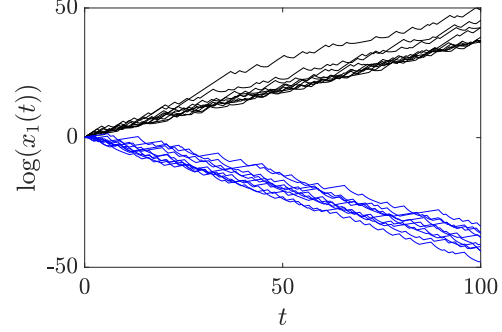


Fig. 3. 10 realizations of $\log x_1(t)$ before and after antibiotic intervention. Black line: $\rho(\mathcal{A}) = 6.28$. Blue line: $\rho(\mathcal{A}) = 0.55$.

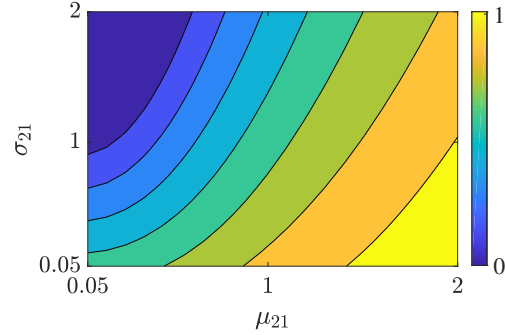


Fig. 4. The ratio of the resource allocated on phenotype 2 versus $\mu_{21} \in [0.01, 2]$ and $\sigma_{21} \in [0.01, 2]$.

the probability density function

$$f(t) = \begin{cases} \gamma_{ij} \frac{1}{x \sigma_{ij} \sqrt{2\pi}} e^{-(\ln x - \mu_{ij})^2 / (2\sigma_{ij}^2)} & 0 < t \leq T, \\ 0, & \text{otherwise.} \end{cases}$$

We truncate this density function at a finite time T to satisfy Assumption 4.4, and γ_{ij} is the constant for normalizing the integral of the truncated density function. Throughout the simulation, we fix $\mu_{12} = 0.2$, $\sigma_{12} = 0.8$, and $T = 100$, while μ_{21} and σ_{21} are subject to changes.

First, we let $\mu_{21} = 0.4$ and $\sigma_{21} = 0.6$ and solve the optimal intervention problem for various values of $c_1 = \{0.2, 0.5, 1\}$ and $s_1^1 \in (0, 2]$ with the budget $\bar{C} = 2$. In Fig. 2, we show the spectral radius of \mathcal{A} under the optimal interventions for each value of c_1 and s_1^1 . We can observe the dependence of the minimized spectral radius on the relevant parameters. For example, when $s_1^1 = 0.2$ and $c_2 = 1$, the optimal intervention reduces the spectral radius $\rho(\mathcal{A})$ from 6.28 to 0.55 with the dose allocations $\alpha_1 = 0.13$, $\alpha_2 = 1.87$. We show the sample paths of the original and the optimized systems in Fig. 3.

We then observe how the shape of the density functions of the sojourn times affects the optimal intervention strategy. Let us fix $c_1 = 1$ and $s_1^1 = 1$ and vary the values of μ_{21} and σ_{21} as $\mu_{21} \in [0.01, 2]$ and $\sigma_{21} \in [0.01, 2]$. For each values of μ_{21} and σ_{21} , we find the optimal amount

of doses α_1^*, α_2^* by using Theorem 1, and compute the ratio $c_2(\alpha_2^*)/C$. We show the computed ratios in Fig. 4. We observe that the variation of μ_{21} and σ_{21} significantly affects the optimal resource allocation pattern. Remarkably, even with the same value of μ_{21} (i.e., even when the average of the sojourn time is fixed), the optimal resource allocation drastically varies dependent on the value of σ_{21} .

5 Conclusion

This paper studied the stabilization problem of positive semi-Markov jump linear systems. By utilizing the spectral property of nonnegative matrices, we proposed a novel computation framework that the optimal performance of the system can be formulated to a convex optimization problem which is solved by optimizing the spectral radius of the matrix under the budget-constrained of the system parameter. Then, we checked the validity through a simulation example of the biological propagation which illustrates the relations among these parameters.

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