Mixed H_2/H_{∞} Control for Markov Jump Linear Systems with State and Mode-observation Delays

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Mixed H_2/H_∞ Control for Markov Jump Linear Systems with State and Mode-observation Delays

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***Abstract*—In this paper, we study state-feedback control of Markov jump linear systems with state and mode-observation delays. We assume that the delay of the mode observation induced by the controllers follows an exponential distribution. We also introduce a time-varying state delay factor that is applied in the state-feedback controller. Our formulation provides a novel framework to analyze and design feedback control laws for the delayed Markov jump linear systems. We present a procedure to transform the closed-loop system into a standard Markov jump linear system with delays. Moreover, based on this transformation, we propose a set of Linear Matrix Inequalities (LMI) to design feedback gains for stabilization and mixed H_2/H_∞ control. Numerical simulation is presented to illustrate the effectiveness of the theoretical results.**

***Index Terms*—Markov jump linear systems, delays, state-feedback control, stabilization, mixed H_2/H_∞ control.**

I. Introduction

Markov jump linear systems are an important subclass of stochastic systems and are usually used to model the system with abrupt variations in their structures, which in part results from the inherent vulnerability of dynamic systems to component failures or repairs, sudden environmental disturbances, changing subsystem interconnections, and abrupt variations of the operating point of a nonlinear plant [1].

Time delays appear in many physical processes and usually lead to poor performance, oscillation, and instability. Thus, the analysis and synthesis of Markov jump linear systems with time delays have received extensive attention. The existing results can be classified into two types: delay-independent and delay-dependent approaches. The references [2], [3], [4] showed that delay-dependent results are generally less conservative than the delay-independent ones, especially when the size of delay is small. One common assumption for most of the above results is that the current system mode and the current system state are detected simultaneously. However, in practice, it inevitably takes some time to identify the system mode and then switch to the corresponding controller. In [5] a class of chemical systems illustrates the necessity of the research of asynchronous switching for efficient control design. The work in [6], [7], [8], [9] investigated the principal control method to let an unstable system remain stable with or without time delays, i.e., construct proper control input signals to achieve the above goal, which requires knowledge of the current system mode and state/output. However, the techniques would be invalid since the difficulty to obtain the information of the current system mode and state/output. To address the above problem, Xiong et al. [10] designed time-delay controllers using the past system information instead of the current system information. Another important type of delay is the random delay of observation process starting from the switching of the mode signal. In this case, the observation signal follows a renewal process. The authors in [11] designed almost-surely state-feedback controllers for stabilization whose gains are reset when an observation is triggered. The authors in [12], [13] devised state-feedback controllers for stabilization when observations are performed periodically. The reference [14] addressed the stabilization problem for single-input Markov jump linear systems via mode-dependent quantized state feedback.

In this paper, we propose a novel framework to analyze and design state-feedback control laws for continuous-time Markov jump linear systems with state and mode-observation delays. Specifically, we assume that the random observation delay from each signal switching follows an exponential distribution. Since in reality we usually do not have efficient methods to obtain the information of the switching signal, it is important to assume that there exists mode-observation delay from the signal. We transform the resulting closed-loop system with state and mode-observation delays into a Markov jump linear system in a standard form. Moreover, based on this transformation, we propose a set of LMIs to design feedback gains for stabilization and mixed H_2/H_∞ control.

This paper is organized as follows. In Section II, we formulate the state-feedback control problem for Markov jump linear systems with state and mode-observation delays. We show in Section III that the resulting closed-loop system can be regarded as a Markov jump linear system by embedding the stochastic process composed by the signal and the observation of the signal into a standard Markov process. In Section IV, we present an LMI formulation to design state-feedback gains for stabilization and mixed H_2/H_∞ control. We examine the effectiveness of our proposed method by a numerical example in Section V.

The notation used in this paper is standard. Let $\mathbb{Z}_+, \mathbb{R}$, and $\mathbb{R}_+$ denote the sets of positive integers, real numbers, positive real numbers, respectively. For $n \in \mathbb{Z}_+$ and $m \in \mathbb{Z}_+$, we let $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the spaces of real n-vectors and $n \times m$ matrices. We use $\|\cdot\|$ to denote the Euclidean norm on $\mathbb{R}^n$. $\Pr(\cdot)$ is used to denote the probability of an event. Expectations are denoted by $E[\cdot]$. Given any real matrix A,

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we use $A^\top$ to denote the transpose of matrix A, $A > 0$ to represent that A is positive-definite, and $\mathrm{Tr}(A)$ to denote the trace of matrix A. When A is symmetric, we let $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and minimal eigenvalue of A, respectively. We let $\mathrm{diag}(\cdot)$ represent a block diagonal matrix. Let I_m denote the $m \times m$ identify matrix. In this paper, we use $\star$ to denote the symmetric blocks of partitioned symmetric matrices. We let $\mathbf{1}$ denote indicator functions. Given $q \in \mathbb{Z}_+$, we denote by $\mathcal{L}_2([0,\infty),\mathbb{R}^q)$ the q-dimensional vector space of all functions $f\colon [0,\infty) \to \mathbb{R}$ satisfying $\int_0^\infty \|f(t)\|^2 dt < \infty$. Given $\tau \in \mathbb{R}_+$, we let $\mathcal{C}([-\tau,0] \to \mathbb{R}^n)$ denote the space of continuous functions mapping $[-\tau,0]$ into $\mathbb{R}^n$.

II. Problem Formulation

In this section, we introduce the problems studied in this paper. Let $r = \{r(t)\}_{t\geq 0}$ be a continuous-time Markov process taking values in $\Theta = \{1,2,\dots,N\}$ and having the following transition probabilities for all $h > 0$ and $i,j \in \Theta$:

$$\Pr(r(t+h) = j \mid r(t) = i) = \begin{cases} \pi_{ij}h + \mathcal{O}(h), & \text{if } i \neq j, \\ 1 + \pi_{ii}h + \mathcal{O}(h), & \text{if } i = j, \end{cases}$$

where $\lim_{h\to\infty} \mathcal{O}(h)/h = 0$ and $\sum_{j\colon j\neq i} \pi_{ij} = -\pi_{ii}$. The initial condition of r is denoted by $r(0) = r_0$. For each $i \in \Theta$, we let $A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times m}, C_i \in \mathbb{R}^{l\times n}, J_i \in \mathbb{R}^{l\times n}, E_i \in \mathbb{R}^{n\times q}, \varPsi_i \in \mathbb{R}^{l\times q}$, and $\varPhi_i \in \mathbb{R}^{l\times q}$ be matrices. Consider the following Markov jump linear system:

$$\Sigma\colon \begin{cases} \dot{x}(t) = A_{r(t)}x(t) + B_{r(t)}u(t) + E_{r(t)}w(t), \\ z(t) = C_{r(t)}x(t) + \varPsi_{r(t)}w(t), \\ y(t) = J_{r(t)}x(t) + \varPhi_{r(t)}w(t), \end{cases}$$

where $w \in \mathcal{L}_2([0,\infty),\mathbb{R}^q)$ is a disturbance input, z is the controlled output, and y is the measurable output.

A. State and mode observation delays

In this paper, we consider the mixed H_2/H_∞ control problem for the Markov jump linear system via delayed state-feedback with a mode-dependent feedback gain. We specifically allow delays in the measurement of both the state variable x and the mode signal r. Therefore, we consider the state-feedback control of the following form:

$$u(t) = K_{\tilde{r}(t)}x(t - \tau(t)), \tag{1}$$

where $K_1,\dots,K_N \in \mathbb{R}^{m\times n}$ are state-feedback gains and τ represents an unknown time-varying delay satisfying

$$\tau(t) \in [0,\tau_0], \ \ \dot{\tau}(t) \in [0,\tau^+]$$

for some positive constants τ_0 and $\tau^+ \in (0,1]$, while $\tilde{r}$ represents a delayed measurement of the mode signal that is described as follows:

1) When the mode signal r changes from i to $j \in \Theta\backslash\{i\}$ at time t, a constant h is drawn from a fixed probability distribution. We call the random variable h the mode observation delay.
2) If r remains to be j until time $t+h$, then the value of $\tilde{r}$ is set to j at time $t+h$.
3) On the other hand, if r changes its state before time $t+h$, then we go back to the first step.

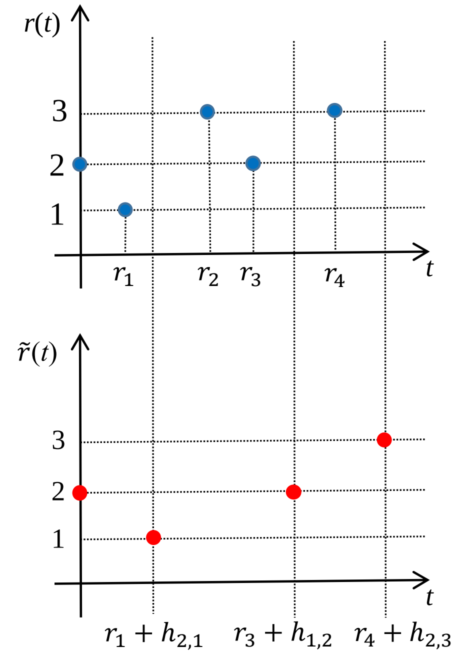

Figure 1: An observation of the mode signal r with the state space $\Theta = \{1,2,3\}$. Let r_α represent the α-th switching of the process $\{r(t)\}_{t\geq 0}$ and h_{j_1,j_2} represent the mode observation delay starting from the most recent r of $\{\tilde{r}(t)\}_{t\geq 0}$ switching from the state j_1 to the state j_2 as shown in this figure, where $\alpha \in \mathbb{Z}_+$, $j_1 \in \Theta$, and $j_2 \in \Theta$. Until the first observation $r_1 + h_{2,1}$, the most recent $\tilde{r}$ is temporarily set to 2.

These properties are illustrated by Fig. 1.

In this paper, we place the following assumption on the mode observation delay.

Assumption 1. *The mode observation delay h from the current observation state i to the correct mode j follows an exponential distribution with rate $g_{ij} > 0$ for all i and $j \in \Theta$.*

B. Problem formulation

The Markov jump linear system Σ and the state-feedback controller (1) yield the following closed-loop system:

$$\Sigma_K\colon \begin{cases} \dot{x}(t) = A_{r(t)}x(t) + B_{r(t)}K_{\tilde{r}(t)}x(t-\tau) + E_{r(t)}w(t), \\ \quad x(\varepsilon) = \phi(\varepsilon), \varepsilon \in [-\tau,0], \\ z(t) = C_{r(t)}x(t) + \varPsi_{r(t)}w(t), \\ y(t) = J_{r(t)}x(t) + \varPhi_{r(t)}w(t), \end{cases}$$

where $\phi \in \mathcal{C}([-\tau,0] \to \mathbb{R}^n)$ is the initial state. The weak delay-dependent stochastic stability of the closed-loop system is defined as follows.

Definition 1. *The system Σ_K is said to be* weakly delay-dependent stochastically stable *if for every $((r_0,\tilde{r}_0),\phi)$ there* *exists a constant $C((r_0,\tilde r_0),\phi) > 0$ such that*

$$E\left[\int_0^\infty \{\|x(t,\phi)\|^2\}dt \mid (r_0,\tilde r_0), x(\varepsilon)=\phi(\varepsilon), \varepsilon\in[-\tau,0]\right] \leq C((r_0,\tilde r_0),\phi)$$

for all $w \in \mathcal{L}_2([0,\infty),\mathbb{R}^q)$.

We then introduce the H_2 and H_∞ performance for the closed-loop system.

Definition 2. *Let a constant $\gamma > 0$ be given. For the system Σ_K, we define the H_2 performance measure as*

$$\mathcal{H}_2 = E\left[\int_0^\infty z^\top(t)z(t)dt\right],$$

and H_∞ performance measure as

$$\mathcal{H}_\infty = E\left[\int_0^\infty \{y^\top(t)y(t) - \gamma^2 w^\top(t)w(t)\}dt\right],$$

respectively.

We now state the problem that we study in this paper.

Problem 1. *Let $f_2 > 0$, $f_\infty > 0$, and $\gamma > 0$ be given. Find state-feedback gains $K_1, \dots, K_N$ such that the closed-loop system Σ_K is weakly delay-dependent stochastically stable and satisfies*

$$\mathcal{H}_2 \leq f_2 \text{ and } \sup_{w\in\mathcal{L}_2([0,\infty),\mathbb{R}^q)} \mathcal{H}_\infty \leq f_\infty. \tag{2}$$

We remark that the closed-loop system Σ_K is not a Markov jump linear system in a standard form since there is a random mode delay in the observation signal. For this reason, we cannot apply the methodologies for the control of Markov jump linear systems available in the literature. In order to overcome this difficulty, in next section we show that the system Σ_K can be transformed into a Markov jump linear system with a standard form.

III. Equivalent Reduction to a Markov Jump Linear System

In this section, we reduce the closed-loop system Σ_K to a Markov jump linear system in a standard form by embedding the stochastic processes in the system Σ_K into an extended Markov process with a larger space. For the system Σ_K, we introduce the notation

$$s(t) = (r(t), \tilde r(t)),$$

which collects the continuous-valued stochastic processes. Also, define $\mathcal{D}$ as the set of $\xi = (i,j) \in \Theta\times\Theta$. The set $\mathcal{D}$ contains all possible values that can be taken by $s = \{s(t)\}_{t\geq 0}$. The initial condition of s is denoted by $s(0) = s_0$.

Let us then show the following proposition, which allows us to rewrite the closed-loop system in terms of a Markov jump linear system in a standard form. The proof of the proposition is omitted because it is a direct consequence from the definition of the observation process $\tilde r$ as well as Assumption 1.

Proposition 1. *The stochastic process s is a time-homogeneous Markov process taking values in $\mathcal{D}$. The transition rate from $\xi = (i_1,j_1)\in\mathcal{D}$ to $\xi' = (i_2,j_2)\in\mathcal{D}$ is given by*

$$q_{\xi\xi'} = \begin{cases} \mathbf{1}(j_1 = j_2)\pi_{i_1 i_2}, & \text{if no observation} \\ \mathbf{1}(i_1 = i_2 = j_2)g_{j_1 j_2,}, & \text{otherwise.}\end{cases}$$

For a family of matrices $\{D_1,\dots,D_N\}$, we use the notations

$$\hat D_{i,j} = D_i$$

and

$$\check D_{i,j} = D_j$$

for all $i,j\in\Theta$. Then, from Proposition 1, we conclude that the closed-loop system Σ_K can be rewritten to the following Markov jump linear system

$$\bar\Sigma_K\colon \begin{cases} \dot x(t) = \hat A_{s(t)}x(t) + \hat B_{s(t)}\check K_{s(t)}x(t-\tau) + \hat E_{s(t)}w(t), \\ \quad x(\varepsilon) = \phi(\varepsilon), \varepsilon\in[-\tau,0], \\ z(t) = \hat C_{s(t)}x(t) + \hat\Psi_{s(t)}w(t), \\ y(t) = \hat J_{s(t)}x(t) + \hat\Phi_{s(t)}w(t), \end{cases}$$

IV. Main Result

We show the main result for designing a mixed H_2/H_∞ controller in this section. Throughout this paper, we fix a one-to-one mapping $\mathcal{F}\colon \Theta\times\Theta \mapsto \{1,\dots,N^2\}$. In the rest of the paper, we use the notation

$$k_{ij} = \mathcal{F}((i,j)),\ \mathcal{F}_0 = \mathcal{F}(s_0)$$

We also let the infinitesimal generator of the stochastic process s be denoted by $\hat{\mathcal{S}} = [\kappa_{kk'}]_{kk'} \in \mathbb{R}^{N^2\times N^2}$. The following theorem allows us to design state-feedback gains having a prescribed H_2 and H_∞ performance measure and is the main result of this section.

Theorem 1. *Let $f_2 > 0, f_\infty > 0, \gamma > 0$ be given constants and $L_k \in \mathbb{R}^{n\times n} > 0$ be given matrices, respectively. The feedback gain*

$$K_j = Z_j Y_j^{-1}$$

is a mixed H_2/H_∞ controller satisfying the performance measure limits in (2) for the closed-loop system Σ_K if there exist matrices $Y_j = Y_j^\top \in \mathbb{R}^{n\times n}$, $\Lambda\in\mathbb{R}$, and $Z_j \in \mathbb{R}^{m\times n}$ satisfying the following system of LMIs:

$$\lambda + \Lambda \leq \min\{f_2, f_\infty\} \tag{3}$$

$$\begin{bmatrix} \mathcal{U}_{k_{ij}} & \star & \star & \star & \star \\ Y_j & -L_{k_{ij}} & \star & \star & \star \\ C_iY_j & 0 & -I & \star & \star \\ Z_j^\top B_i^\top & 0 & 0 & 0 & \star \\ \aleph_{k_{ij}}^\top & 0 & 0 & 0 & -\mathcal{Y}_{k_{ij}} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathcal{U}_{k_{ij}} & Y_j & Y_jJ_i^\top & B_iZ_j & \aleph_{k_{ij}} & E_i + Y_jJ_i^\top\Phi_i \\ \star & -L_{k_{ij}} & 0 & 0 & 0 & 0 \\ \star & \star & -I & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & -\mathcal{Y}_{k_{ij}} & 0 \\ \star & \star & \star & \star & \star & -\gamma^2 I + \Phi_i^\top\Phi_i \end{bmatrix} < 0,$$

$$\begin{bmatrix} -\lambda & \star \\ \phi(0) & -Y_j \end{bmatrix} \leq 0, \begin{bmatrix} -\Lambda & \star \\ X & -\frac{1}{\lambda_{\max}(L_k^{-1})} \end{bmatrix} \leq 0, Y_j > 0,\ \Lambda > 0$$

where the matrices $\mathcal{U}_k$, $\mathcal{Y}_k$, $\aleph_k$, and X are defined by

$$\begin{aligned}
\mathcal{U}_{k_{ij}} &= Y_j A_i^\top + A_i Y_j + \kappa_{k_{ij}k_{ij}} Y_j,\\
\mathcal{Y}_{k_{ij}} &= \operatorname{diag}(Y_1, \ldots, Y_{k_{ij}-1}, Y_{k_{ij}+1}, \ldots, Y_{N^2}),\\
\aleph_{k_{ij}} &= [\sqrt{\kappa_{k_{ij}1}}Y_j \ldots \sqrt{\kappa_{k_{ij}(k_{ij}-1)}}Y_j, \sqrt{\kappa_{k_{ij}(k_{ij}+1)}}Y_j \ldots \\
&\quad \sqrt{\kappa_{k_{ij}N^2}}Y_j],\\
X &= \left(\int_{-\tau}^{0} x^\top(t)x(t)dt \right)^{\frac{1}{2}}.
\end{aligned}$$

A. Proof

In this subsection, we present the proof of Theorem 1. We show the following two propositions for designing a mixed H_2/H_∞ controller for the system Σ_K, in which the first proposition provides the condition of a H_2 controller with limited H_2 performance measure.

Proposition 2. *Let $Q_k \in \mathbb{R}^{n\times n}$ be a positive definite matrix. Assume that there exist positive definite matrices $P_k \in \mathbb{R}^{n\times n}$ satisfying*

$$\Xi_{k_{ij}} = \begin{bmatrix} \eth_{k_{ij}} & \star \\ K_j^\top B_i^\top P_{k_{ij}} & -\hat{Q}_{k_{ij}} \end{bmatrix} < 0 \tag{4}$$

for all $(i,j) \in \Theta \times \Theta$, where

$$\eth_{k_{ij}} = A_i^\top P_{k_{ij}} + P_{k_{ij}} A_i + Q_{k_{ij}} + \sum_{k'=1}^{N^2} \kappa_{k_{ij}k'} P_{k'} + C_i^\top C_i$$

and $\hat{Q}_{k_{ij}} = (1-\tau^+)Q_{k_{ij}}$. If $w(t) \equiv 0$, then

$$\mathcal{H}_2 \leq x_0^\top P_{\mathcal{F}_0} x_0 + \int_{-\tau}^{0} \phi^\top(v) Q_{\mathcal{F}_0} \phi(v) dv. \tag{5}$$

Proof. We let the Lyapunov function $V(\cdot)$ be

$$\begin{aligned} V(x(t),t,k) &= x^\top(t)P_k x(t) + \int_{t-\tau}^{t} x^\top(v)Q_k x(v)dv,\\ k &= \mathcal{F}(s(t)). \end{aligned} \tag{6}$$

By [15], [16], we have that the weak infinitesimal operator $\mathcal{S}^x[\cdot]$ of the process $\{x(t)\}_{t\geq 0}$ in the system Σ_K is

$$\mathcal{S}^x[V] = \frac{\partial V}{\partial t} + \dot{x}^\top(t)\frac{\partial V}{\partial x}\big|_k + \sum_{k'=1}^{N^2} \kappa_{kk'} V(x(t),t,k).$$

We define

$$\begin{aligned} \tilde{\Xi}_{k_{ij}} = A_i^\top P_{k_{ij}} + P_{k_{ij}} A_i + Q_{k_{ij}} + \sum_{k'=1}^{N^2} \kappa_{k_{ij}k'} P_{k'} \\ + P_{k_{ij}} B_i K_j \hat{Q}_{k_{ij}}^{-1} K_j^\top B_i^\top P_{k_{ij}}. \end{aligned}$$

Let

$$\bar{\Xi}_{k_{ij}} = \tilde{\Xi}_{k_{ij}} + C_i^\top C_i$$

for each k_{ij}, we see that $\bar{\Xi}_{k_{ij}} < 0$ is equivalent with (4). From the proof of Theorem 3.1 in [17], we have

$$\mathcal{S}^x[V] \leq x^\top(t)\tilde{\Xi}_k x(t). \tag{7}$$

Beside, the fact $C_i^\top C_i > 0$ and the LMIs (4) lead to

$$\tilde{\Xi}_k < 0 \tag{8}$$

for all $(i,j) \in \Theta \times \Theta$.

Since $\|x(t+\alpha)\|^2 \leq \psi\|x(t)\|^2$ for all $\alpha \in [-\tau, 0]$ and some $\psi > 0$, we conclude that

$$V(x(t),t,k) \leq x^\top(t)P_k x(t) + \mu\|x\|^2 \tag{9}$$

by (6), where $\mu = \psi\tau\lambda_{\max}(Q_k)$. Thus, from (7), (8), and (9) we obtain

$$\begin{aligned} \frac{\mathcal{S}^x[V]}{V(x(t),t,k)} &\leq \frac{x^\top(t)\tilde{\Xi}_k x(t)}{x^\top(t)P_k x(t) + \mu\|x\|^2} \\ &\leq -\min_{k\in\{1,\ldots,N^2\}} \left\{ \frac{\lambda_{\min}(-\tilde{\Xi}_k)}{\lambda_{\max}(P_k) + \mu} \right\} \\ &= -\zeta < 0 \end{aligned}$$

for all $x \neq 0$, which induces

$$\mathcal{S}^x[V] \leq -\zeta V(x(t),t,k). \tag{10}$$

In accordance with Dynkin's formula

$$\begin{aligned} &E[V(x(t),t,k)] - V(x_0, 0, \mathcal{F}_0) \\ &\qquad = E\Big[\int_0^t \mathcal{S}^x[V(x(v),v,\mathcal{F}(s(v)))]dv\Big], \end{aligned} \tag{11}$$

we obtain

$$\frac{dE[V(x(t),t,k)]}{dt} = E\big[\mathcal{S}^x[V]\big] \leq -\zeta V(x(t),t,k),$$

which results in

$$E[V(x(t),t,k)] \leq e^{-\zeta t} V(x_0, 0, \mathcal{F}_0). \tag{12}$$

From (10), (11), and (12), we have that

$$\begin{aligned} &\sup_{T\in[0.\infty)} \Big[E\big[V(x(T),T,\mathcal{F}(s(T)))\big] - V(x_0,0,\mathcal{F}_0) \Big] \\ &\quad = (e^{-\zeta T} - 1)V(x_0,0,\mathcal{F}_0) \\ &\quad \leq -\zeta E\Big[\int_0^T V(x(t),t,\mathcal{F}(s(t)))dt\Big], \end{aligned}$$

should be established, where $e^{-\zeta T} - 1 \in (-1, 0]$.

It can be readily shown that $E[\int_{-\tau}^{0} x^\top(t+\alpha)Q_k x(t+\alpha)d\alpha] \geq 0$. Therefore,

$$\begin{aligned} &\lim_{T\to\infty} E\Big[\int_0^T x^\top(t)x(t)dt \mid (x_0, s_0)\Big] \\ &\leq \lim_{T\to\infty} E\Big[\int_0^T V(x(t),t,\mathcal{F}(s(t)))dt \mid (x_0,s_0)\Big] \\ &\leq x_0^\top \lambda_{\max}(\Omega_k)x_0 \\ &< \infty, \end{aligned}$$

where

$$\begin{aligned} &\Omega_k \\ &= \max_k \left\{ \frac{P_{\mathcal{F}_0}\|x_0\|^2 + \tau_0\lambda_{\max}(Q_{\mathcal{F}_0})\big[\sup\|x(t+\alpha)\|^2\big]}{\zeta\lambda_{\max}(P_k)\|x_0\|^2} \right\}. \end{aligned}$$

This illustrates that the system Σ_K is weakly delay-dependent stochastically stable with the controller (1). Now, let us show the upper bound of H_2 performance measure. By (7) and Dynkin's formula (11), we have

$$\begin{aligned}&E\big[V\big(x(\Gamma),\Gamma,\mathcal{F}(s(\Gamma))\big)\big]-V(x_0,0,\mathcal{F}_0)\\&\leq E\bigg[\int_0^{\Gamma}x^\top(v)\Big(\bar{\Xi}_{\mathcal{F}(s(v))}-C_{r(v)}^\top C_{r(v)}\Big)x(v)dv\bigg],\end{aligned}$$

Then, we can show that

$$\begin{aligned}\mathcal{H}_2&=E\bigg[\int_0^{\infty}z^\top(v)z(v)dv\bigg]\\&=E\bigg[\int_0^{\infty}\big[x^\top(v)C_{r(v)}^\top C_{r(v)}x(v)\big]dv\bigg]\\&\leq V(x_0,0,\mathcal{F}_0)\\&=x_0^\top P_{\mathcal{F}_0}x_0+\int_{-\tau}^{0}\phi^\top(v)Q_{\mathcal{F}_0}\phi(v)dv\end{aligned}$$

is established if (4) holds true. This completes the proof. □

The following proposition provides an LMI formulation to guarantee the weak delay-dependent stochastic stability and γ attenuation property of a H_∞ controller, as well as an upper bound of the H_∞ performance measure of the H_∞ controller.

Proposition 3. *Let $\gamma>0$ be a given constant. Let $Q_k\in\mathbb{R}^{n\times n}$ be a positive definite matrix. Assume that there exist positive definite matrices $P_k\in\mathbb{R}^{n\times n}$ satisfying the following LMIs*

$$\hat{\Xi}_{k_{ij}}=\begin{bmatrix}\Re_{k_{ij}} & \star & \star\\ K_j^\top B_i^\top P_{k_{ij}} & -\hat{Q}_{k_{ij}} & \star\\ E_i^\top P_{k_{ij}}+\Phi_i^\top J_i & 0 & -\gamma^2 I+\Phi_i^\top\Phi_i\end{bmatrix}<0 \tag{13}$$

for all $(i,j)\in\Theta\times\Theta$, where the matrix $\Re_{k_{ij}}$ is defined by

$$\Re_{k_{ij}}=A_i^\top P_{k_{ij}}+P_{k_{ij}}A_i+Q_{k_{ij}}+\sum_{k'=1}^{N^2}\kappa_{k_{ij}k'}P_{k'}+J_i^\top J_i.$$

Then, the closed-loop system Σ_K is weakly delay-dependent stochastically stable and satisfies

$$\mathcal{H}_\infty\leq x_0^\top P_{\mathcal{F}_0}x_0+\int_{-\tau}^{0}\phi^\top(v)Q_{\mathcal{F}_0}\phi(v)dv, \tag{14}$$

for all $w\in\mathcal{L}_2([0,\infty),\mathbb{R}^q)$.

Proof. We have already shown the stochastic stability with $w(t)\equiv 0$ in the proof of Proposition 2. Then let us show that the system is with a disturbance attenuation level γ. We have already defined

$$V(x(t),t,k)=x^\top(t)P_kx(t)+\int_{t-\tau}^{t}x^\top(v)Q_kx(v)dv.$$

Let

$$\mathcal{S}_x[V]=\mathcal{S}^x[V]+x^\top(t)P_kE_iw(t)+w^\top(t)E_i^\top P_kx(t).$$

Applying Dynkin's formula (11), we can show that

$$z^\top(t)z(t)-\gamma^2w^\top(t)w(t)+\mathcal{S}_x[V]\leq\mathcal{X}^\top(t)\hat{\Xi}_k\mathcal{X}(t) \tag{15}$$

by Σ_K and (7), where

$$\mathcal{X}^\top(t)=[x^\top(t)\ x^\top(t-\tau)\ w^\top(t)].$$

The inequality (15) can be transformed into

$$\begin{aligned}\mathcal{H}_T\leq E\bigg\{\int_0^T\mathcal{X}^\top(t)\hat{\Xi}_k\mathcal{X}(t)dt\bigg\}\\-E\big\{V(x(t),t,k)\mid_{t=T}+V(x_0,0,\mathcal{F}_0)\big\}.\end{aligned} \tag{16}$$

Based on the LMIs (13) and $E\{V(x(t),t,k)\mid_{t=T}\}\geq 0$, inequality (16) leads to

$$\mathcal{H}_\infty=\mathcal{H}_{T\to\infty}\leq V(x_0,0,\mathcal{F}_0).$$

This completes the proof of Proposition 3. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. We have that the LMIs (4) in Proposition 2 are equivalent to

$$\bar{\Xi}_{k_{ij}}<0$$

for all k_{ij}. Using $Y_j=P_j^{-1}$ and Y_j to pre-multiply and to post-multiply $\bar{\Xi}_{k_{ij}}$, respectively, we obtain the following inequalities

$$\begin{aligned}&Y_jA_i^\top+A_iY_j+(B_iZ_jY_j^{-1})\hat{Q}_{k_{ij}}^{-1}(Y_j^{-\top}Z_j^\top B_i^\top)\\&+Y_jL_{k_{ij}}^{-1}Y_j+Y_jC_i^\top C_iY_j+Y_j\bigg(\sum_{k'=1}^{N^2}\kappa_{k_{ij}k'}P_{k'}\bigg)Y_j<0,\end{aligned} \tag{17}$$

where $L_{k_{ij}}=Q_{k_{ij}}^{-1}$ for each $k_{ij}\in\{1,\ldots,N^2\}$. The inequalities (17) are equal to

$$\begin{bmatrix}\mathcal{U}_{k_{ij}} & \star & \star & \star & \star\\ Y_j & -L_{k_{ij}} & \star & \star & \star\\ C_iY_j & 0 & -I & \star & \star\\ Y_j^{-\top}Z_j^\top B_i^\top & 0 & 0 & -\hat{Q}_{k_{ij}} & \star\\ \aleph_{k_{ij}}^\top & 0 & 0 & 0 & -\mathcal{Y}_{k_{ij}}\end{bmatrix}<0. \tag{18}$$

From (18) we see that

$$\begin{bmatrix}\mathcal{U}_{k_{ij}} & \star & \star & \star & \star\\ Y_j & -L_{k_{ij}} & \star & \star & \star\\ C_iY_j & 0 & -I & \star & \star\\ Z_j^\top B_i^\top & 0 & 0 & 0 & \star\\ \aleph_{k_{ij}}^\top & 0 & 0 & 0 & -\mathcal{Y}_{k_{ij}}\end{bmatrix}+\begin{bmatrix}0 & \star & \star\\ 0 & -Y_j^\top\hat{Q}_{k_{ij}}Y_j & \star\\ 0 & 0 & 0\end{bmatrix}<0,$$

which is the condition ensuring that the controller (1) is a H_2 controller. Since $\hat{Q}_{k_{ij}}=(1-\tau^+)Q_{k_{ij}}\geq 0$, we have

$$\begin{bmatrix}0 & \star & \star\\ 0 & -Y_j^\top\hat{Q}_{k_{ij}}Y_j & \star\\ 0 & 0 & 0\end{bmatrix}\leq 0.$$

Therefore, if the second LMIs of (3) are satisfied for all $(i,j)\in\Theta\times\Theta$, then the controller (1) is a H_2 controller for the system Σ_K. Using the same argument in the above part of this proof, we can also prove that the third LMIs of (3) are proposed for a H_∞ controller design as presented in Proposition 3.

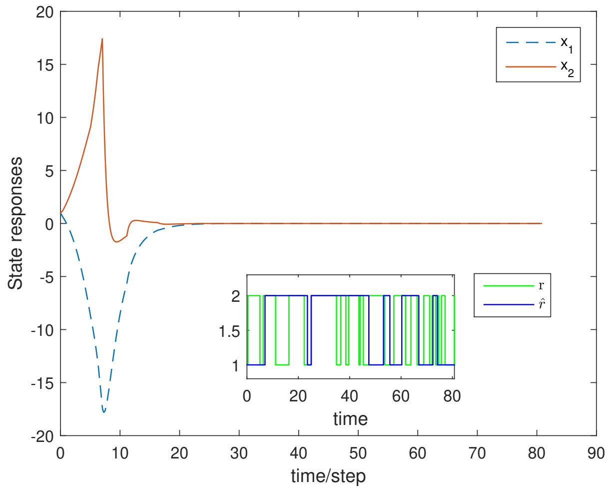

Figure 2: State trajectories

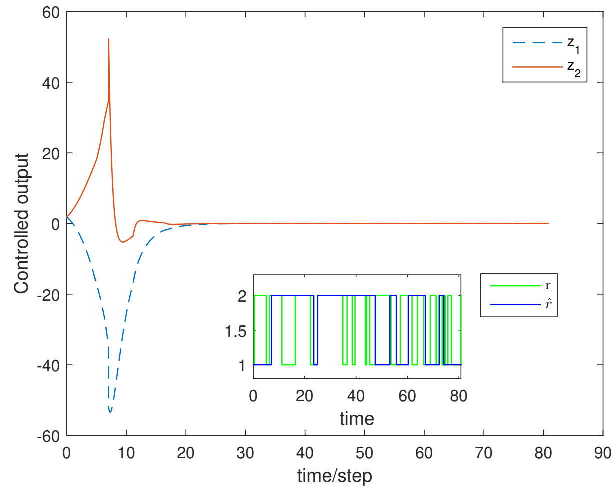

Figure 3: Controlled output $z(t)$

We then show the first inequality of (3). Let

$$\phi^\top(0)P_{\mathcal{F}_0}\phi(0) \le \max\left\{\phi^\top(0)P_1\phi(0),\ldots,\phi^\top(0)P_{N^2}\phi(0)\right\} = \lambda,$$

and $Y_j^{-1} = P_j$ for each $j \in \Theta$. Then,

$$-\lambda + \phi^\top(0)Y_j^{-1}\phi(0) \le 0, \phi(0) = x_0.$$

Moreover,

$$\begin{aligned} &\sup \int_{-\tau}^{0} x^\top(s)Q_{\mathcal{F}_0}x(s)ds \\ &= \lambda_{\max}(Q_{\mathcal{F}_0})\left[\int_{-\tau}^{0} x^\top(s)x(s)ds\right] \\ &= X^\top\lambda_{\max}(Q_{\mathcal{F}_0})X. \end{aligned}$$

We assume that there exists a real number $\Lambda > 0$ such that

$$X^\top\lambda_{\max}(L_k^{-1})X \le \Lambda,$$

then

$$-\Lambda + X^\top\lambda_{\max}(L_k^{-1})X \le 0.$$

Therefore, if there exist λ and Λ such that

$$x_0^\top P_{\mathcal{F}_0}x_0 + \int_{-\tau}^{0} x^\top(s)Q_{\mathcal{F}_0}x(s)ds \le \lambda + \Lambda$$

holds true, then we obtain

$$\mathcal{H}_2 \le \lambda + \Lambda, \text{and } \mathcal{H}_\infty \le \lambda + \Lambda$$

by (5) and (14). If $\lambda + \Lambda \le \min\{f_2, f_\infty\}$ holds, we have that (2) is satisfied. □

V. Numerical Example

In this section, we illustrate the effectiveness of Theorem 1 by a numerical example. We let $\Theta = \{1, 2\}$ in this example. We use $h_{1,2}$ and $h_{2,1}$ to represent the observation delay of $\tilde{r}$ from state 1 to state 2, and the one from state 2 to state 1, respectively. We assume that both $h_{1,2}$ and $h_{2,1}$ follow the exponential distribution with rate value 3. Let the infinitesimal generator of the Markov process r be

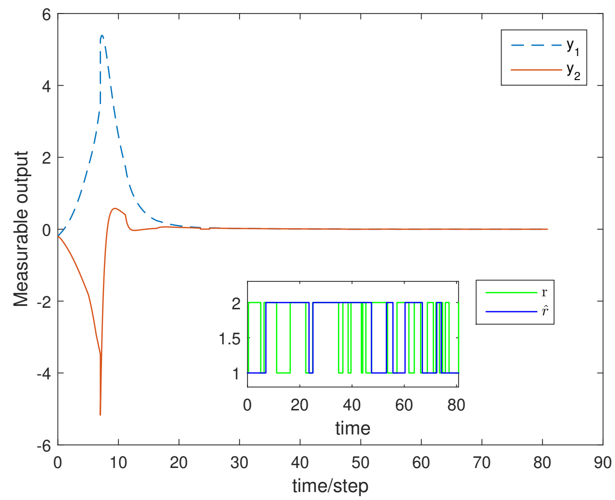

Figure 4: Measurable output $y(t)$

$$\begin{bmatrix} -5 & 5 \\ 3 & -3 \end{bmatrix},$$

so that

$$\tilde{\mathcal{S}} = \begin{bmatrix} -5 & 0 & 5 & 0 \\ 3 & -8 & 0 & 5 \\ 3 & 0 & -6 & 3 \\ 0 & 3 & 0 & -3 \end{bmatrix}.$$

The system matrices are given by

$$A_1 = \begin{bmatrix} 0 & -0.45 \\ 0.9 & 0.9 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & -0.29 \\ 0.9 & -1.26 \end{bmatrix}, B_1 = \begin{bmatrix} 0.5 \\ 1.1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.6 \\ 1.4 \end{bmatrix}, E_1 = \begin{bmatrix} 0 & -0.01 \\ -0.01 & -0.03 \end{bmatrix}, E_2 = \begin{bmatrix} -0.01 & -0.03 \\ -0.06 & -0.1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, C_2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \Psi_1 = \begin{bmatrix} 0.4 & 0.5 \\ -0.3 & 1.2 \end{bmatrix},$$

$$\Psi_2 = \begin{bmatrix} -0.2 & -0.4 \\ 0 & -0.6 \end{bmatrix}, J_1 = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.2 \end{bmatrix},$$

$$J_2 = \begin{bmatrix} -0.3 & 0 \\ 0 & -0.3 \end{bmatrix}, \Phi_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \Phi_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Regarding the mixed H_2/H_∞ control, we let

$$\tau^+ = 0.5, \gamma = 1, f_2 = 15, f_\infty = 17,$$

$$Q_1 = Q_2 = Q_3 = Q_4 = I_2,$$

$$w(t) = [0.5e^{-0.1t}\sin(0.01\pi t) \quad 0.5e^{-0.1t}\sin(0.01\pi t)]^\top,$$

so that $\hat{Q}_k = (1-\tau^+)Q_k = 0.5I_2$ for all $k \in \{1, \dots, 4\}$. Also, we set the initial state $\phi(0) = [0 \ \ 1]^\top$ and $X = 2$. The simulation result shows that there exist

$$Y_1 = \begin{bmatrix} 0.1109 & -0.0541 \\ -0.0541 & 0.2465 \end{bmatrix}, Y_2 = \begin{bmatrix} 0.0702 & -0.0433 \\ -0.0433 & 0.1729 \end{bmatrix},$$

$$Z_1 = [-0.0603 \quad -0.0603]^\top, Z_2 = [-0.0209 \quad -0.0209]^\top,$$

$$\lambda = 7.1444, \min\{\Lambda\} = 4,$$

such that the LMIs of (3) are established. Thus, we have

$$K_1 = \begin{bmatrix} -0.7423 & -0.4074 \end{bmatrix}, K_2 = \begin{bmatrix} -0.4397 & -0.2309 \end{bmatrix}.$$

Finally, we present the curve of state trajectories, controlled output, and measurable output of the system Σ_K against t in Fig 2, Fig 3, and Fig 4, respectively.

VI. Conclusion

In this paper, we have proposed a general framework to analyze and design state feedback control methods for Markov jump linear system with state and mode-observation delays. The mode-observation delay is assumed to follow an exponential distribution. We have first shown that the resulting closed-loop system can be transformed into a Markov jump linear system in a standard form. Based on this transformation, we have proposed an LMI framework to design feedback gains for stabilization and mixed H_2/H_∞ control. Finally, we have examined the effectiveness of our proposed framework by a numerical example.

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